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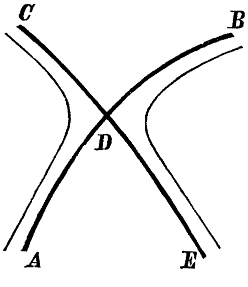
Forms of Non-Singular Quintic Curves.

By David Barcroft.

Before noticing the forms of the curves given in this paper, I shall make a general statement of the principles on which their construction is based, together with some facts concerning inflections and bitangents of curves in general when derived by the method here considered. The construction of the curves is founded on the possibility of deriving one curve from another by a change of parameters. Here the only parametric variation employed is one involving the absolute term alone of the equation representing the curve, which, in every case, is assumed to be an improper curve composed of two non-singular curves. The equation of such a curve is $uv = 0$, and that of the derived curve $uv = k$. If k is properly chosen, the derived curve will be non-singular, and will be as near as we please to the original. This fact enables us to determine its form if we know the character of the changes brought about by a parametric variation of the kind in question. What these changes are can be readily determined.



Let D be a double point of $uv = 0$. For any particular value of k the product uv must preserve the same sign, namely, the sign of k; u and v, therefore, must change together if they change at all. A simultaneous change can take place only when we pass from an angle ADC to its opposite angle BDE.

If the sign of k be changed, parts of the curve will be in the opposite angles ADE and CDB.

To investigate the nature of $uv = k$ as to the direction of its points relative to $uv = 0$, let x and y be the coordinates of any point on $uv = 0$, and ε_1 and ε_2 the increments of x and y necessary to give a point on $uv = k$; substituting $x + \varepsilon_1$ for x, and $y + \varepsilon_2$ for y, in $uv = 0$, we have

$$\left(u + \varepsilon_1 \frac{\partial u}{\partial x} + \varepsilon_2 \frac{\partial u}{\partial y} + \text{etc.}\right)\left(v + \varepsilon_1 \frac{\partial v}{\partial x} + \varepsilon_2 \frac{\partial v}{\partial y} + \text{etc.}\right) = k.$$

If the point (x, y) is on only one of the curves $u = 0$, $v = 0$, say $u = 0$, leaving out higher powers of ε_1 and ε_2, the result becomes

$$\left(\varepsilon_1 \frac{\partial u}{\partial x} + \varepsilon_2 \frac{\partial u}{\partial y}\right) v = k,$$

the equation of a line parallel to the tangent to the curve $u = 0$ at the point (x, y). This line gives the direction of an element of the curve $uv = k$. If (x, y) is a double point on $uv = 0$, our result will be, under the conditions prescribed above,

$$\left(\varepsilon_1 \frac{\partial u}{\partial x} + \varepsilon_2 \frac{\partial u}{\partial y}\right)\left(\varepsilon_1 \frac{\partial v}{\partial x} + \varepsilon_2 \frac{\partial v}{\partial y}\right) = k,$$

which shows that in the neighborhood of a double point of $uv = 0$, $uv = k$ approximates to an hyperbola whose asymptotes are the tangents at the double point in question.

In the cases of the cubic and quartic, the maximum number of real inflections is one-third of the total number. The conic having no inflections, may be included as possessing the same property. All the quintics whose forms I give have fifteen real inflections, and it is true generally that the maximum number of real inflections of curves formed as these are is one-third of the total number. Let it be granted that the number of real inflections of a curve $u = 0$ of degree n_1 is $n_1(n_1 - 2)$, and of a curve $v = 0$ of degree n_2 is $n_2(n_2 - 2)$; let $u = 0$ and $v = 0$ intersect in the maximum number of points $n_1 n_2$, then the curve $uv = k$, k being small enough, will have $(n_1 + n_2)(n_1 + n_2 - 2)$ real inflections; for the inflections of $uv = k$ are those due to the inflections of $uv = 0$ together with two for each double point of $uv = 0$; that is, $n_1(n_1 - 2) + n_2(n_2 - 2) + 2n_1 n_2$, which equals the number just given. In view of the statement made regarding the conic and cubic, the property noticed is seen to be true generally. We may state, therefore, that a curve of any degree n may have as many as $n(n - 2)$

real inflections. It will be observed that, as the intersections of $u = 0$ and $v = 0$ diminish in sets of two, the number of real inflections will diminish in sets of four; hence, if we form a complete system of curves by this method, the number of inflections of the curves of various classes will be $n(n-2)$, $n(n-2)-4$, $n(n-2)-8$, etc.

The bitangents of $uv = k$ may be considered as derived from those of $uv = 0$, including among the latter the analytical bitangents; these are (a), lines drawn from a double point of $uv = 0$ tangent to the curve; (b), lines joining the double points two and two. Each of the analytical bitangents of class (a) gives rise to two, and each of class (b) to four bitangents of $uv = k$; each ordinary bitangent of $uv = 0$ is the source of one of $uv = k$. A figure will make this clear. It will be observed that bitangents derived from analytical bitangents may be imaginary. The total number is made up of

$$\frac{1}{2}(n^4 - 2n^3 - 9n^2 + 18n) - 2n_1n_2\left[n_1(n_1-1) - n_1n_2 - 5\right],$$

derived from the ordinary bitangents of $uv = 0$; $2n_1n_2\left[n(n-1) - 2n_1n_2 - 4\right]$, derived from the analytical bitangents of class (a), and $2n_1n_2(n_1n_2 - 1)$ from those of class (b), making together $\frac{1}{2}(n^4 - 2n^3 - 9n^2 + 18n)$, which is the number for a non-singular curve of degree n. n_1 and n_2 represent the orders of $u = 0$ and $v = 0$.

Zeuthen has remarked (Math. Annalen, Vol. 7, p. 410) that the branches that make up a plane curve may be of two kinds, odd or even; an odd branch intersecting any other odd branch in an odd number of points, and an even branch intersecting any other branch, odd or even, in an even number of points. Möbius has shown that an odd branch has always an odd number of real inflections and an even branch an even number, and if an odd branch is non-singular, it must have at least three real inflections. Applying these statements to the quintic, it is seen that a non-singular quintic cannot have more than one odd branch. It must have one with at least three real inflections. It cannot have more than six even branches external to each other, for, if it had as many as seven, we could pass a cubic through seven points chosen on these and two on the odd branch intersecting the quintic in at least seventeen points. There cannot be more than two even branches if one is inside of the other, and, moreover, the internal branch must be an oval.

Zeuthen divides bitangents into two kinds. His bitangents of the first kind are those whose points of contact are on the same branch; the points of contact may be imaginary. Bitangents of the second kind have their points of contact on different branches. In the case of quartics, the points of contact of a bitangent of the first kind cannot be separated by its intersections with two other bitangents. This is true of quintics if the bitangent has its points of contact on an even branch, but the contrary may be the case if the points of contact are on the odd branch. We may have an inflectional tangent, tangent elsewhere. The number of bitangents of the first kind to an even branch is easily seen to be one-half of the number of its real inflections. Quintics may have 0, 1, 2, 3, 4, 5 or 6 even branches, hence the number of bitangents of the second kind to even branches may be 0, 0, 4, 12, 24, 40, or 60, since every combination of even branches in sets of two has four bitangents of the second kind.

The quintics whose forms are given are derived in every case from the combination of a cubic and conic intersecting in six points. Such combinations give curves with fifteen real inflections, and they illustrate, to a certain extent, the forms of quintics having a less number of inflections, especially those resulting from cubics and conics intersecting in four, two, and zero points. In the following I give forms so far as I have been able to determine them. After Zeuthen an even branch is called an oval, a unifolium, a bifolium, a trifolium, etc., according as it has 0, 2, 4, 6, etc. $2r$ real inflections.

In the following enumeration of forms, I put in the same class quintics having the same number of real inflections, then subdivide according to the number of inflections on the even branches. The odd branches are not mentioned, but must be understood as forming a part of the quintic in every case. Each figure represents two forms, in some cases duplicates. To distinguish between them they have been drawn in heavy and light lines. The dotted curves indicate the combination of cubic and conic from which particular forms are derived. The positions of the inflections on the cubics are shown by the dashes drawn across them.

Quintics with Fifteen Real Inflections.

I.

1. Sexifolium and internal oval. Figs. 1, 4.
2. Six unifolia. Figs. 1, 4.

II.

3. A quadrifolium and a unifolium. Fig. 32.
4. A quadrifolium, a unifolium, and an oval. Fig. 33.
5. A trifolium and a bifolium. Fig. 5.
6. A trifolium, a bifolium, and an oval. Fig. 34.
7. A quintifolium and two ovals. Fig. 8.
8. A bifolium and three unifolia. Figs. 2, 9, 13, 14, 17.
9. A bifolium, three unifolia, and an oval. Figs. 22, 31.
10. Two bifolia, a unifolium, and two ovals. Fig. 23.
11. Five unifolia. Figs. 24, 26, 27.

III.

12. A quadrifolium and an oval. Figs. 35, 44.
13. A quadrifolium and two ovals. Fig. 15.
14. A trifolium and a unifolium. Figs. 11, 12.
15. A trifolium, a unifolium, and an oval. Figs. 18, 36.
16. Two bifolia. Fig. 16.
17. Two bifolia and an oval. Fig. 37.
18. Two bifolia and two ovals. Figs. 19, 28.
19. A bifolium and two unifolia. Figs. 38, 39, 41, 42, 45.
20. Four unifolia. Figs. 3, 10, 20.
21. A quadrifolium and three ovals.
22. A bifolium, two unifolia, and an oval.

The figures for 21 and 22 are not given; 21 results from a combination of a bipartite redundant hyperbola with a hyperbola; 22 from the same cubic with an ellipse; Figs. 41, 42, 45 illustrate this case.

IV.

23. A trifolium and internal oval. Fig. not drawn. The form differs from the next only in the internal oval.
24. A trifolium. Figs. 40, 43, 46.
25. A trifolium and an oval. Fig. 40.
26. A trifolium and two ovals. Figs. 7, 39.

27. A trifolium and three ovals. Fig. 12.
28. A bifolium and a unifolium. Fig. 6.
29. A bifolium, a unifolium, and an oval. Figs. 25, 29, 30.
30. Three unifolia. Figs. 25, 29, 30.
31. Three unifolia and an oval. Fig. 11.

V.

32. A bifolium. Figs. 7, 8, 15, 22, 23, 24, 26, 27, 31, 44.
33. A bifolium and an oval. Figs. 2, 14, 21.
34. Two unifolia. Fig. 47.
35. A bifolium and three ovals.
36. Two unifolia and two ovals.

The last two may be obtained by combining a bipartite redundant hyperbola with an ellipse.

VI.

37. A unifolium and an oval. Figs. 3, 10, 20.
38. A unifolium and three ovals.

The last is formed as 35 and 36 by changing the position of the ellipse. The bipartite cubic corresponding to that of Fig. 43 is the proper one for the cases 35, 36, 38.

VII.

39. An oval. Figs. 18, 33, 34, 36.
40. Two ovals. Figs. 9, 13, 16, 17, 41, 42, 44, 45.
41. Three ovals. Figs. 40, 43, 46.
42. Four ovals.

By changing the position of the ellipse in case 38, 42 is obtained.

43. No even branches. Figs. 28, 47.

Quintics with Eleven Real Inflections.

I.

1. A quadrifolium and an internal oval.
2. A quadrifolium and an oval.
3. Two bifolia and an oval.
4. Four unifolia.
5. A trifolium and a unifolium.
6. A trifolium, a unifolium, and an oval.

II.

7. A trifolium.
8. A trifolium and an oval.
9. A trifolium and two ovals.
10. A trifolium and three ovals.
11. A bifolium and a unifolium.
12. A bifolium, a unifolium, and an oval.
13. A bifolium, a unifolium, and two ovals.
14. Three unifolia.
15. Three unifolia and an oval.

III.

16. A bifolium and an internal oval.
17. A bifolium and an oval.
18. A bifolium.
19. Two unifolia.

IV.

20. A unifolium and an oval.

V.

21. An oval.
22. Two ovals.
23. Three ovals.
24. No even branches.

Quintics with Seven Real Inflections.

I.

1. A bifolium with internal oval.
2. A bifolium and an oval.
3. A bifolium.
4. Two unifolia.

II.

5. A unifolium with internal oval.
6. A unifolium and an oval.
7. A unifolium.

III.

8. An oval.
9. Two ovals.
10. No even branches.

Quintics with Three Real Inflections.

I.

1. An oval with internal oval.
2. Two ovals.
3. An oval.

The total number of forms enumerated is eighty.

An inspection of the figures will show that the odd branches are composed of one, three, or five infinite parts with a number of inflections varying from three to fifteen. It will be observed that in the neighborhood of a double point of the improper curve there are always two inflections of the non-singular curve; they are both in the same angle if the parts of the improper curve turned towards each other are convex; one in each of two opposite angles if the parts turned towards each other are, one concave and the other convex. Other inflections are those in the neighborhood of the inflections of the cubic.

With regard to the real bitangents I am unable to state a general law as to their number. It is certain, however, that the number of bitangents of the first kind is not constant. An examination of the curves will show this to be the case. Figs. 13 and 14, 26 and 27 illustrate the loss of two bitangents of the first kind. Intermediate between the heavy curve of Fig. 13 and the light curve of Fig. 14 there is one having an inflectional bitangent. The loss of two bitangents occurs in passing through this case. It will be noticed that two bitangents of the second kind are lost in the case of the other curves of these figures. The light curves of Figs. 6, 7, and 8, and the heavy curve of Fig. 5, illustrate the statement made that the points of contact of a bitangent of the first kind to an odd branch may be separated by the intersections of the bitangent with two others.

I add the following table showing the character of the even branches as they occur in the figures. 0 represents an oval; 1, a unifolium; 2, a bifolium, etc.; (0) indicates that the oval is inside of another even branch. By a proper combination of digits the number and characters of the even branches are shown in each case. *h* refers to the heavy curve and *l* to the light one.

Figs.	1	2	3	4	5	6
h	6, (0);	2, 0;	1, 0;	6, (0);	0, 0;	2, 1;
l	1, 1, 1, 1, 1, 1;	2, 1, 1, 1;	1, 1, 1, 1;	1, 1, 1, 1, 1, 1;	3, 2;	0, 0;

Figs.	7	8	9	10	11	12
h	2;	5, 0, 0;	2, 1, 1, 1;	1, 0;	3, 1;	3, 1;
l	3, 0, 0;	2;	0, 0;	1, 1, 1, 1;	1, 1, 1, 0;	3, 0, 0, 0;

Figs.	13	14	15	16	17	18
h	2, 1, 1, 1;	2, 0;	2;	0, 0;	2, 1, 1, 1;	0;
l	0, 0;	2, 1, 1, 1;	4, 0, 0;	2, 2;	0, 0;	3, 1, 0;

Figs.	19	20	21	22	23	24
h	2, 2, 0, 0;	1, 0;	2, 0;	2;	2;	2;
l	2, 2, 0, 0;	1, 1, 1, 1;	3, 0;	2, 1, 1, 1, 0;	2, 2, 1, 0, 0;	1, 1, 1, 1, 1;

Figs.	25	26	27	28	29	30
h	2, 1, 1;	2;	2;		2, 1, 0;	2, 1, 0;
l	1, 1, 1;	1, 1, 1, 1, 1;	1, 1, 1, 1, 1;	2, 2, 0, 0;	1, 1, 1;	1, 1, 1;

Figs.	31	32	33	34	35	36
h	2;	4, 1;	0;	0;	4, 0;	3, 1, 0;
l	2, 1, 1, 1, 0;	4, 1;	4, 1, 0;	3, 2, 0;	4, 0;	0;

Figs.	37	38	39	40	41	42
h	2, 2, 0;	2, 1, 1;	3, 0, 0;	3;	0, 0;	0, 0;
l	2, 2, 0;	2, 1, 1;	2, 1, 1;	0, 0, 0;	2, 1, 1;	2, 1, 1;

Figs.	43	44	45	46	47
h	0, 0, 0;	4, 0;	0, 0;	3;	
l	3;	2;	2, 1, 1;	0, 0, 0;	1, 1.

I wish to state, in conclusion, that I am indebted to Dr. Story, of the Johns Hopkins University, for most of the matter of the preceding pages. A number of the curves also were pointed out by him, and, in carrying out the work, I have derived much assistance from his suggestions.

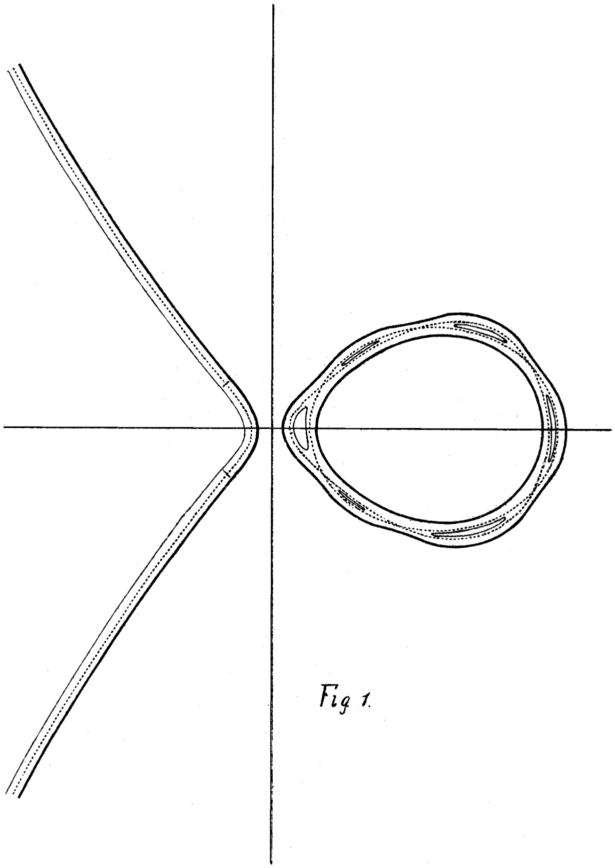


Fig. 1.

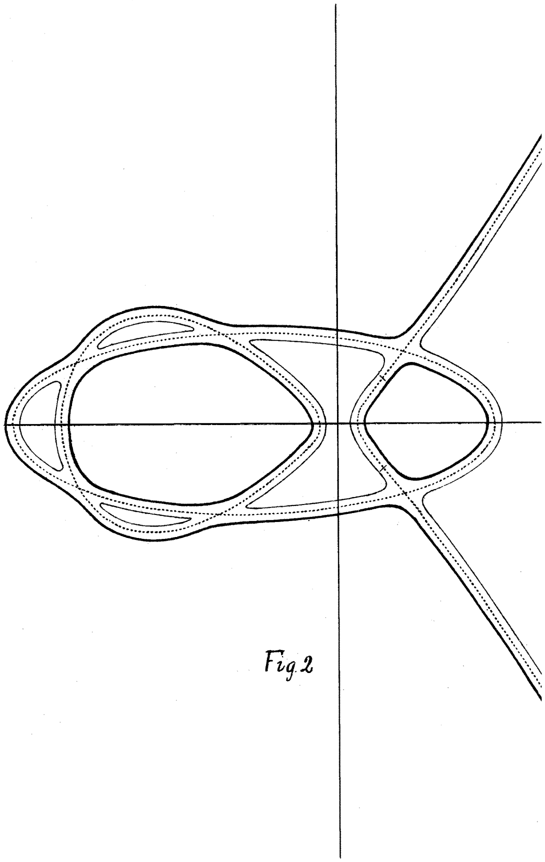


Fig. 2

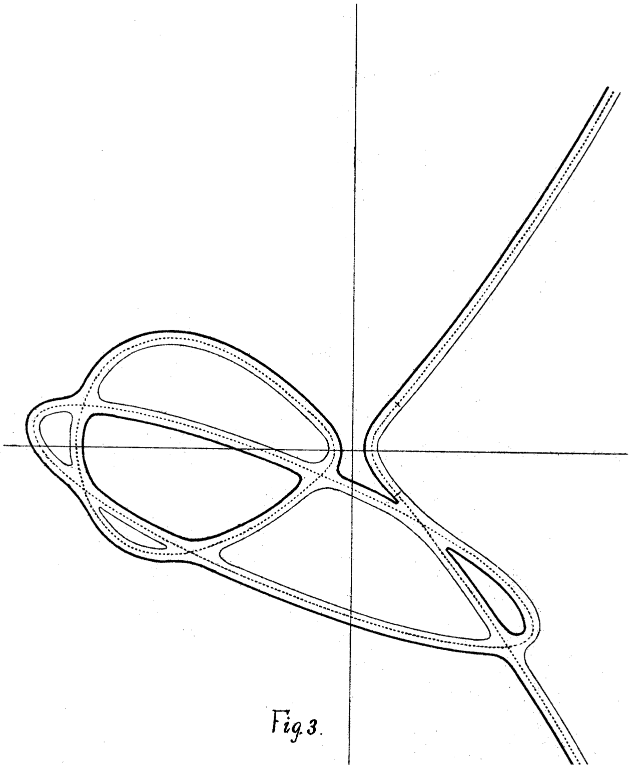


Fig. 3.

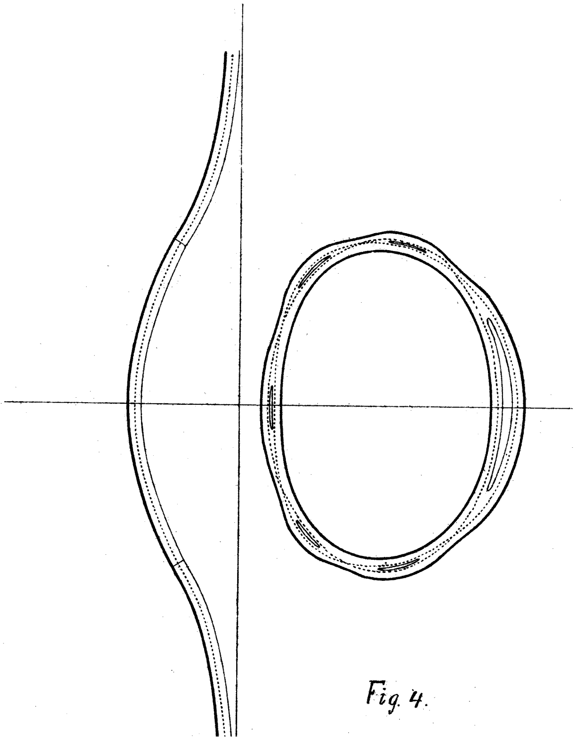


Fig. 4.

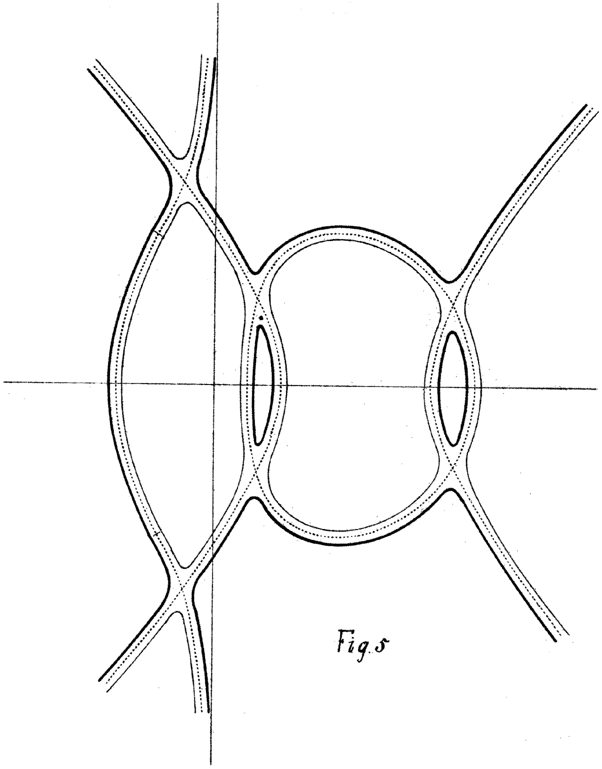


Fig. 5

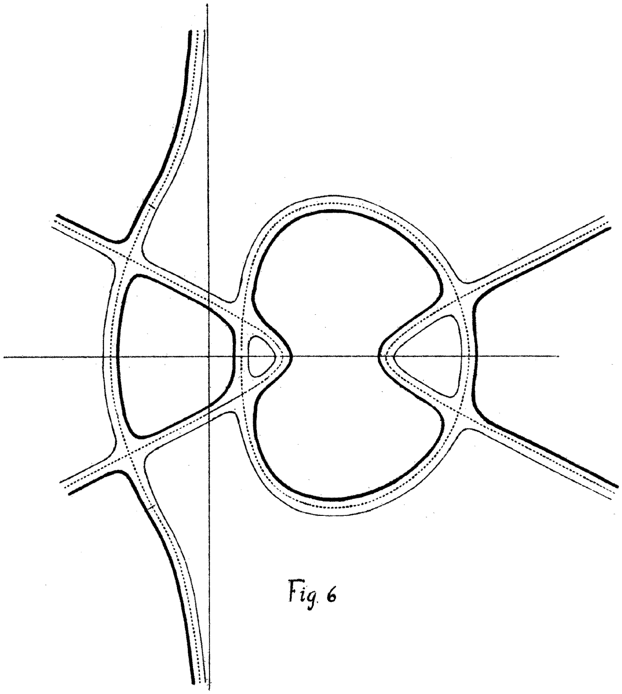


Fig. 6

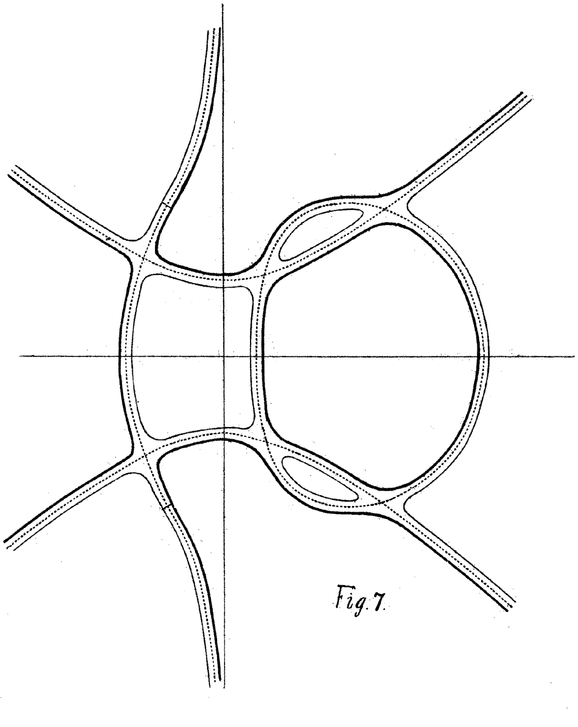


Fig. 7.

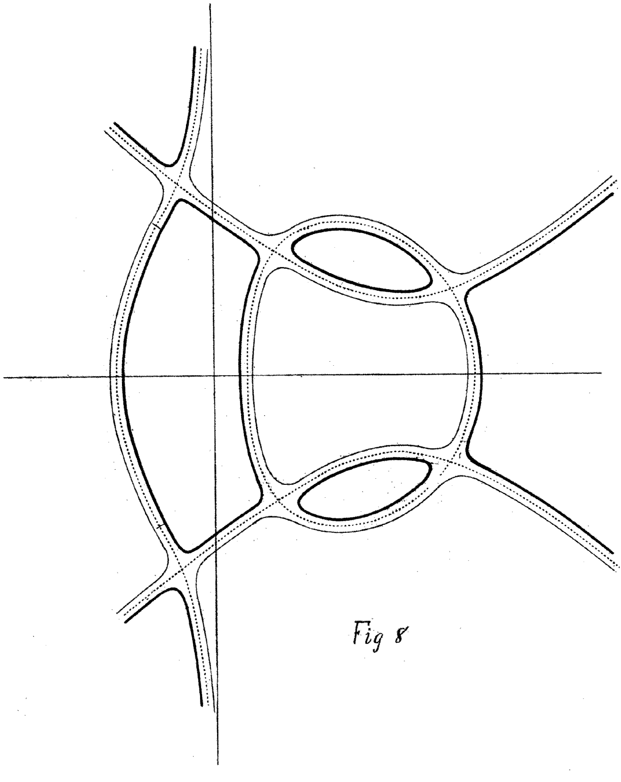


Fig 8

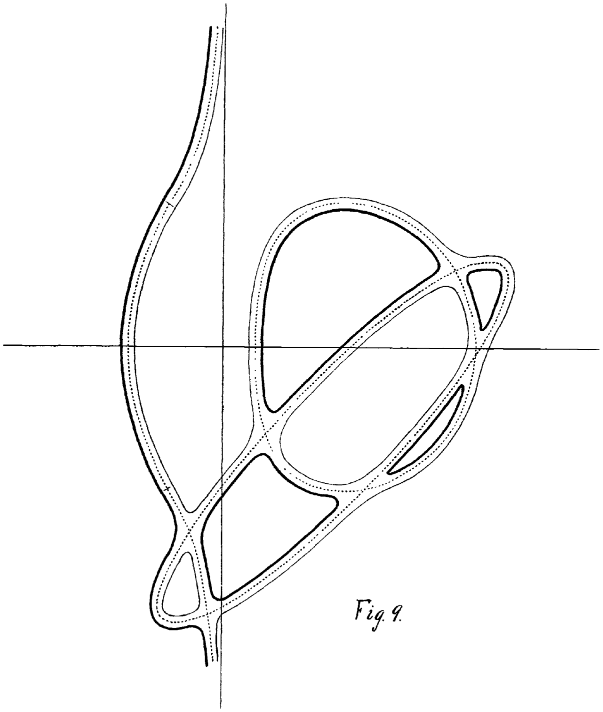


Fig. 9.

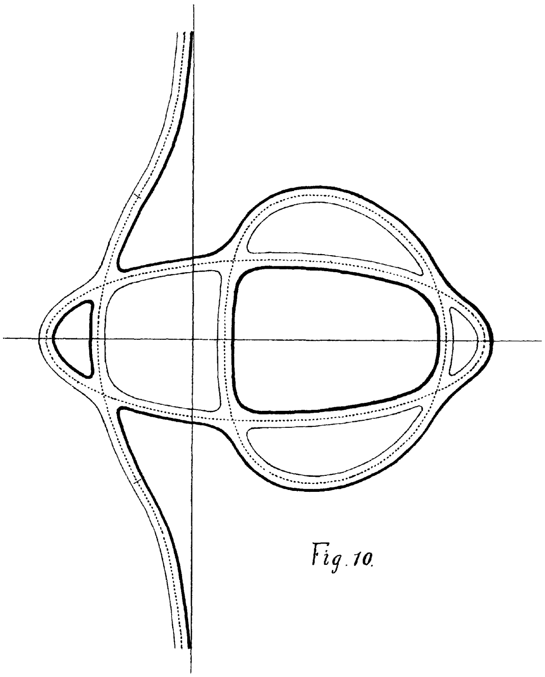


Fig. 10.

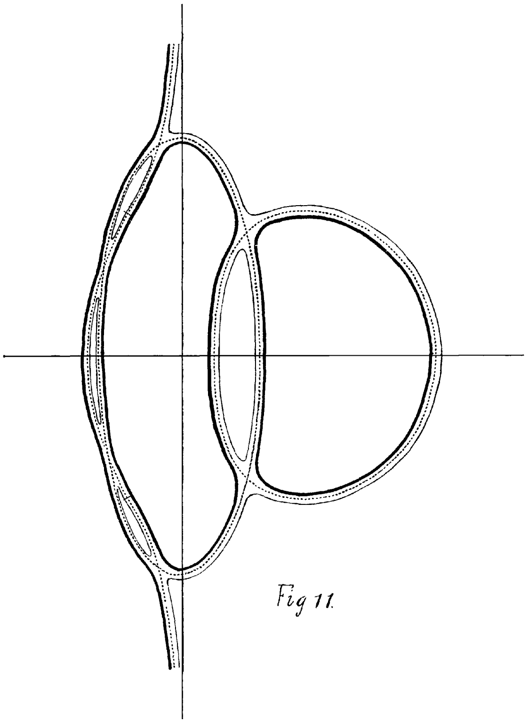


Fig 11.

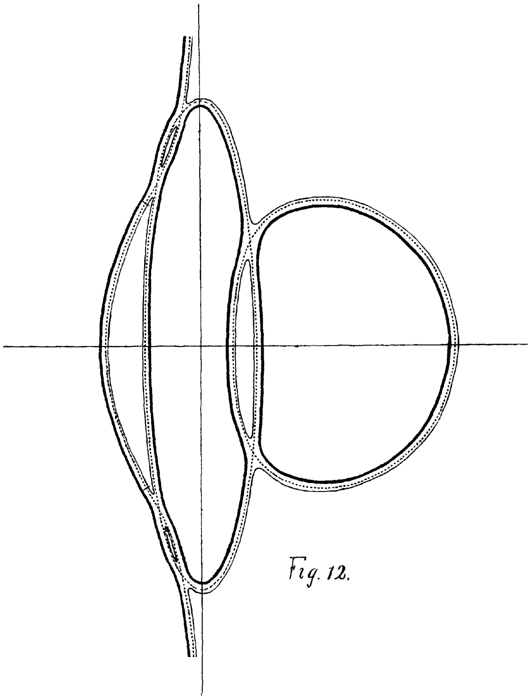


Fig. 12.

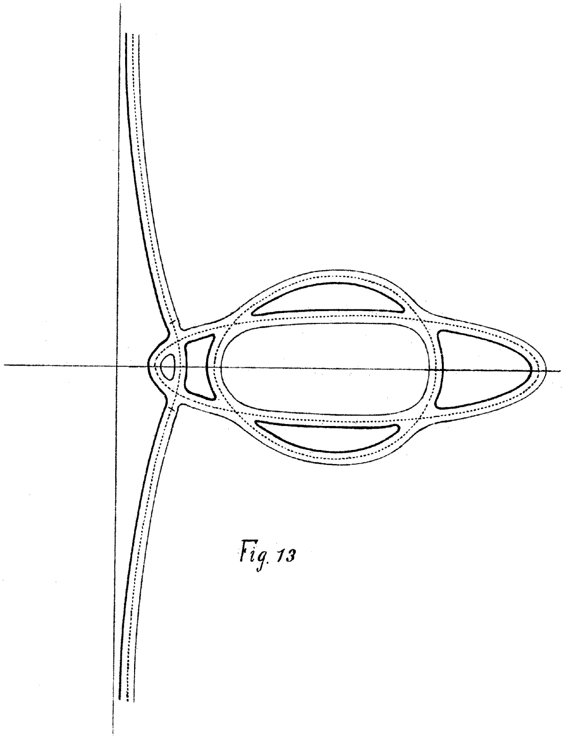


Fig. 13

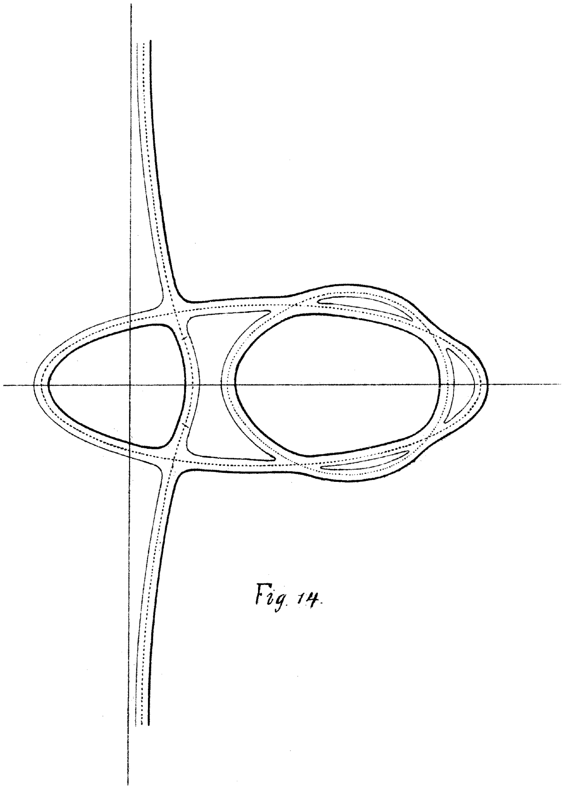


Fig. 14.

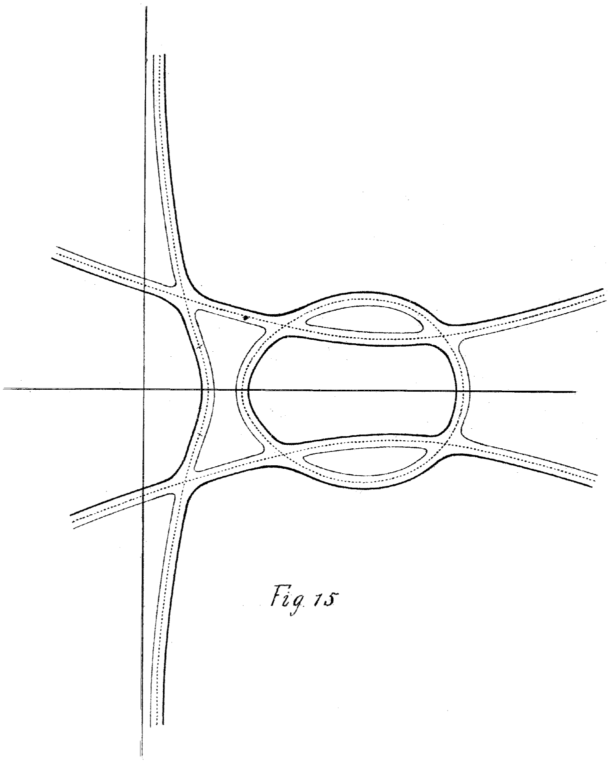


Fig. 15

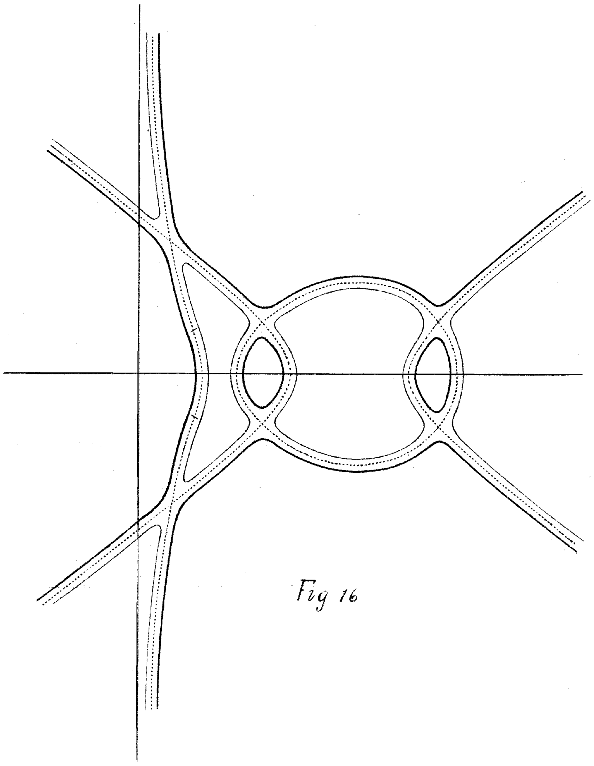


Fig 16

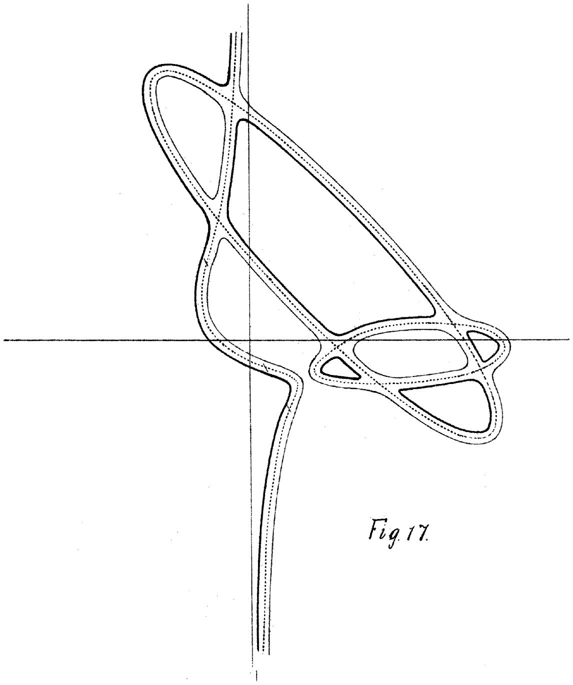
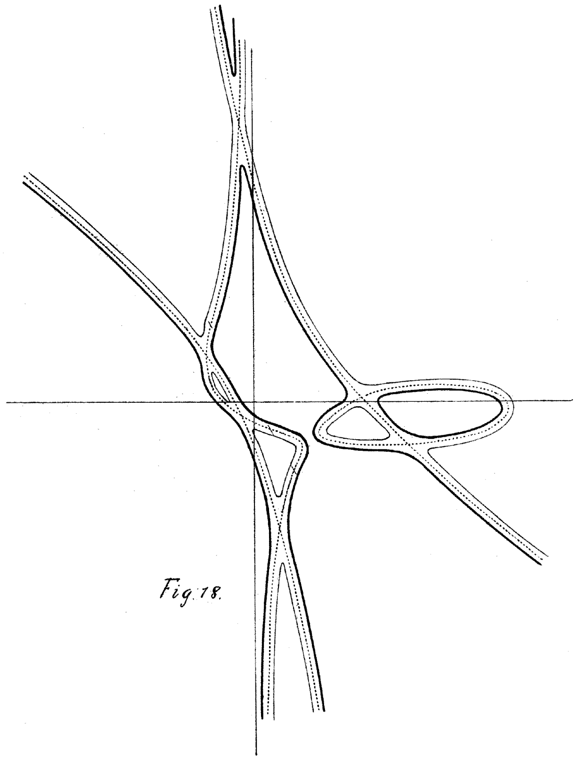


Fig. 17.



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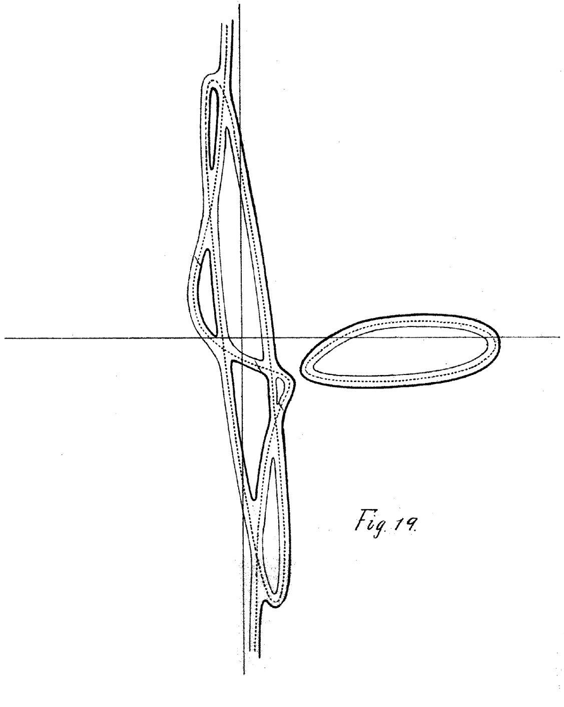
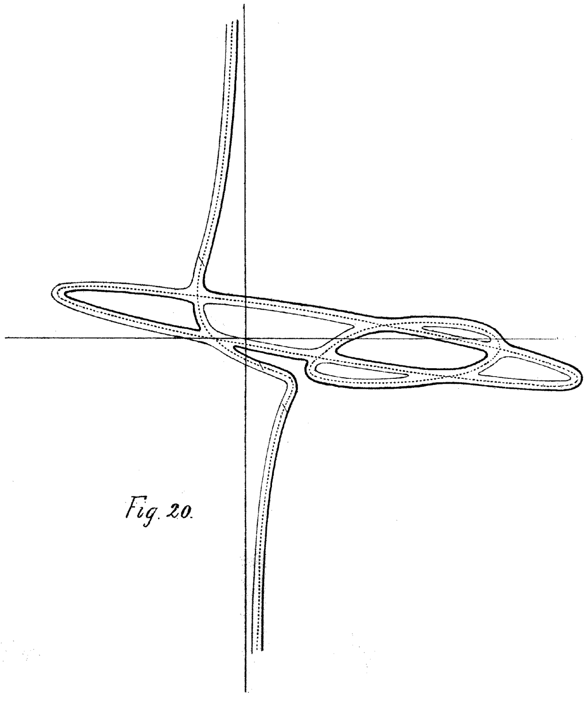


Fig. 19.



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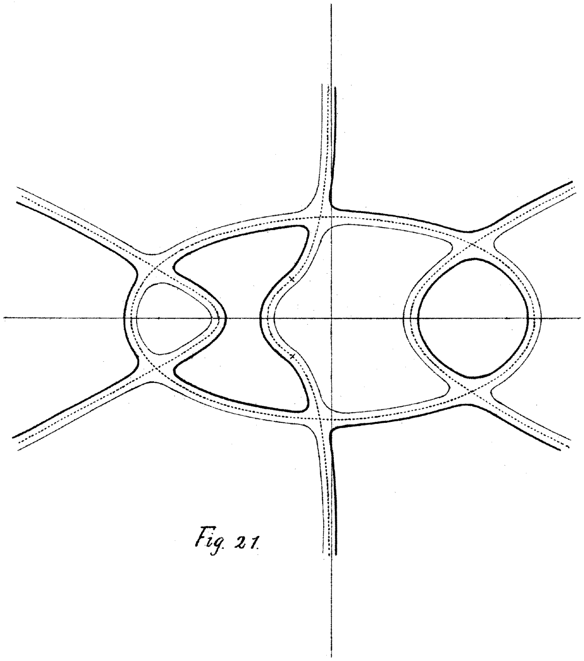


Fig. 21.

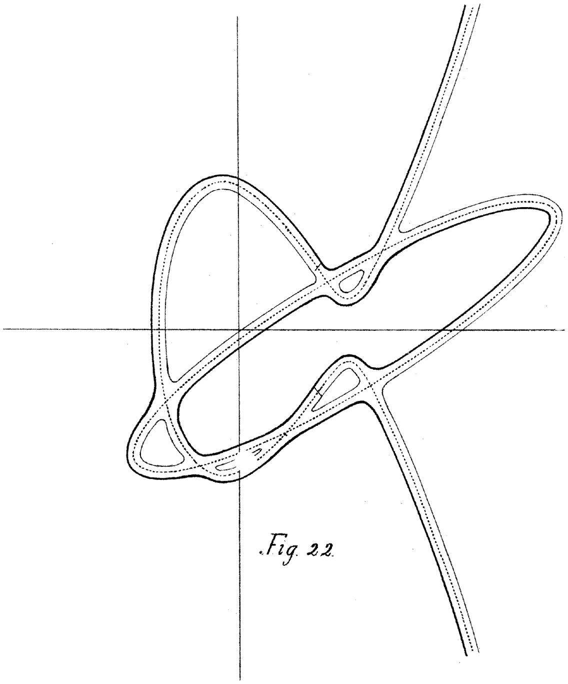


Fig. 22.

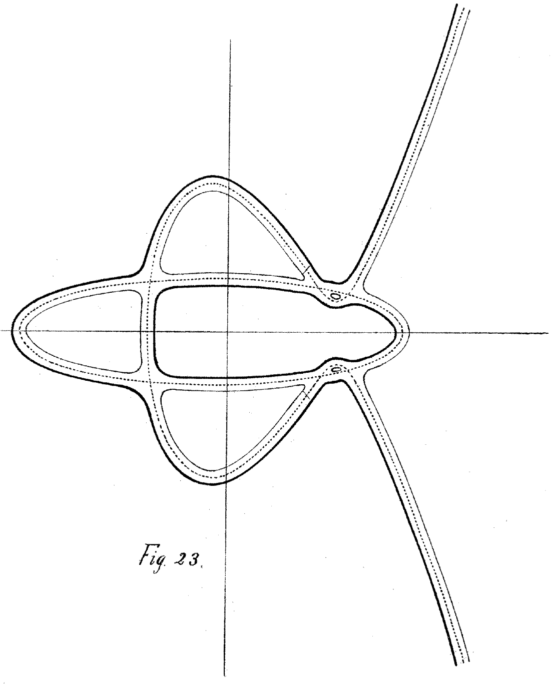


Fig. 23.

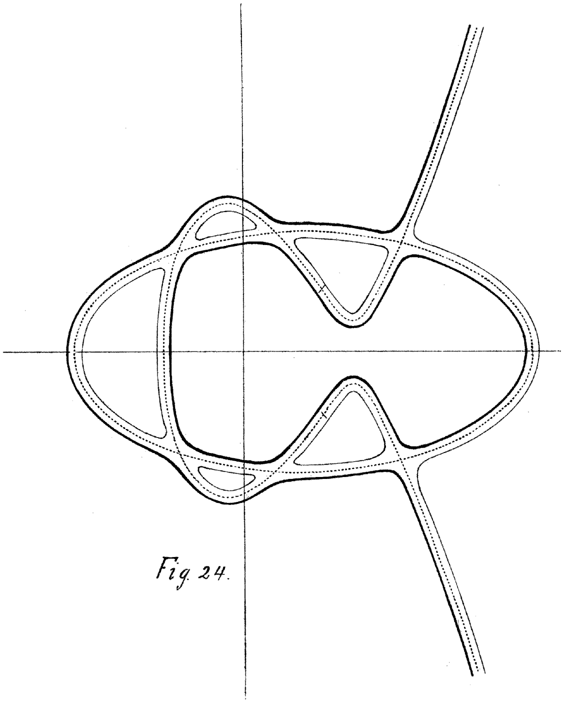
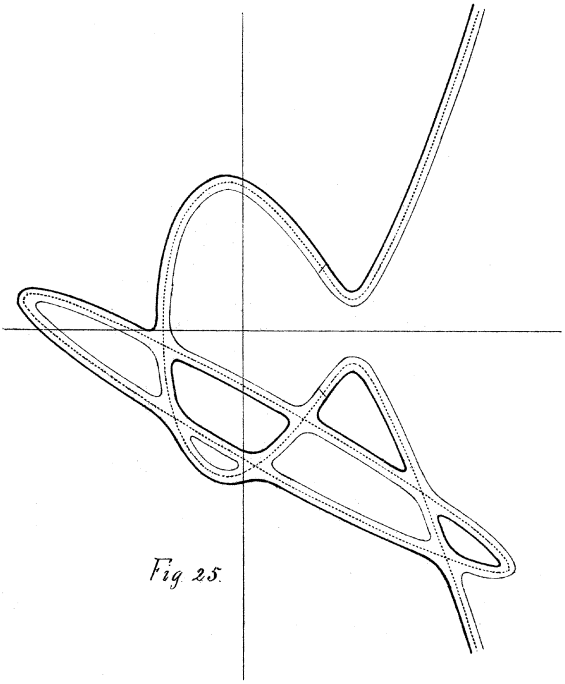


Fig. 24.



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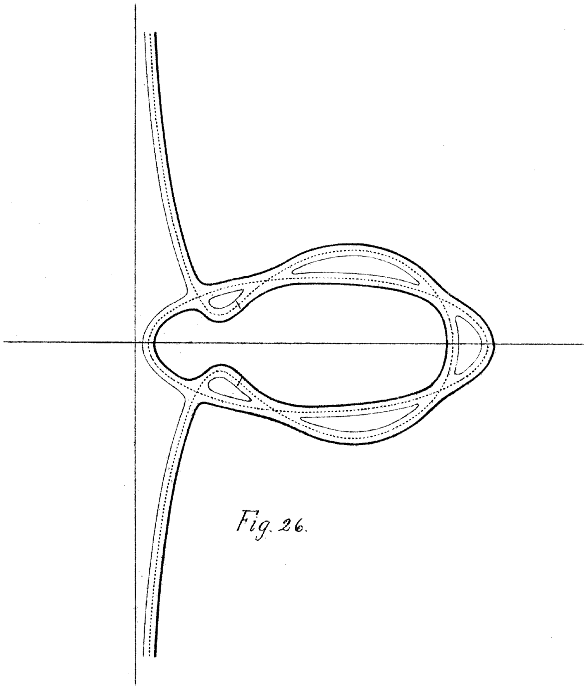

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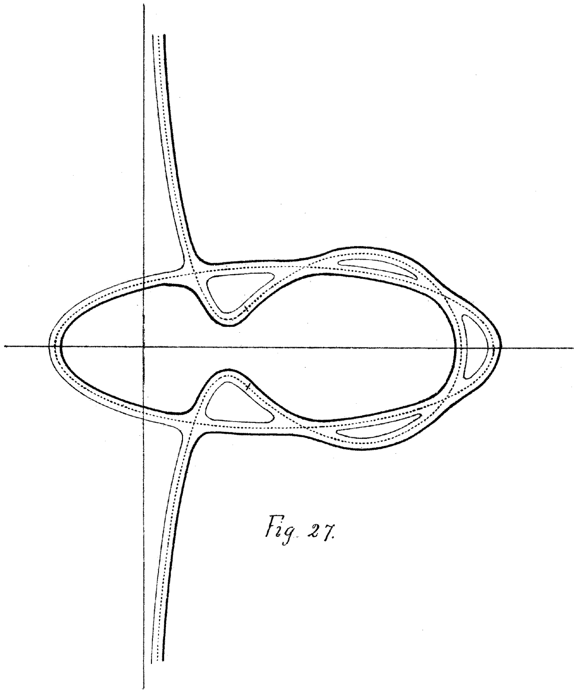

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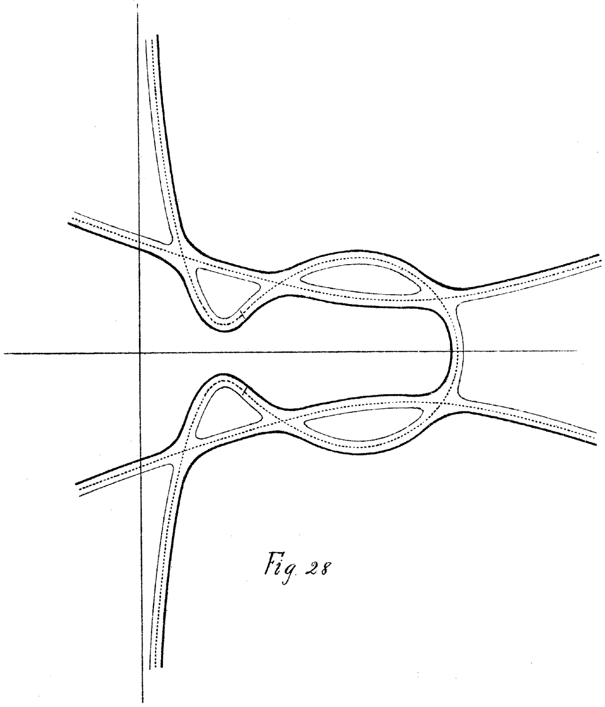

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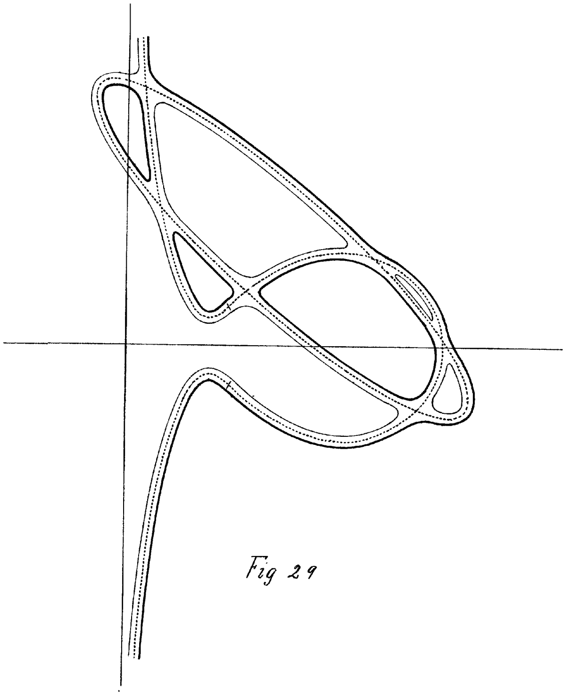

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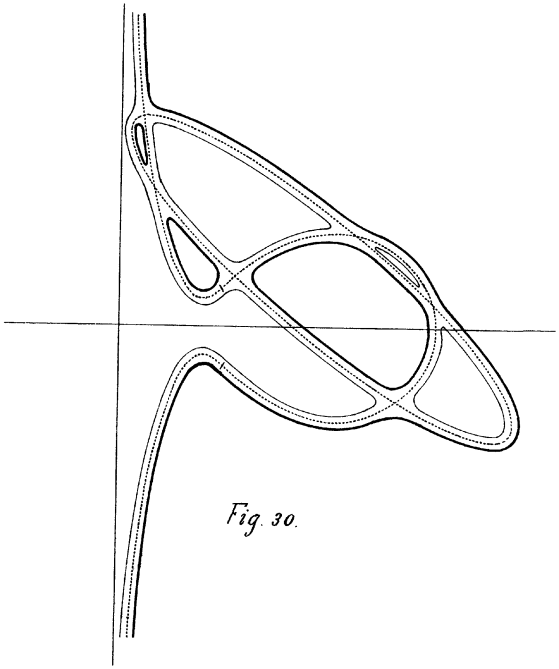

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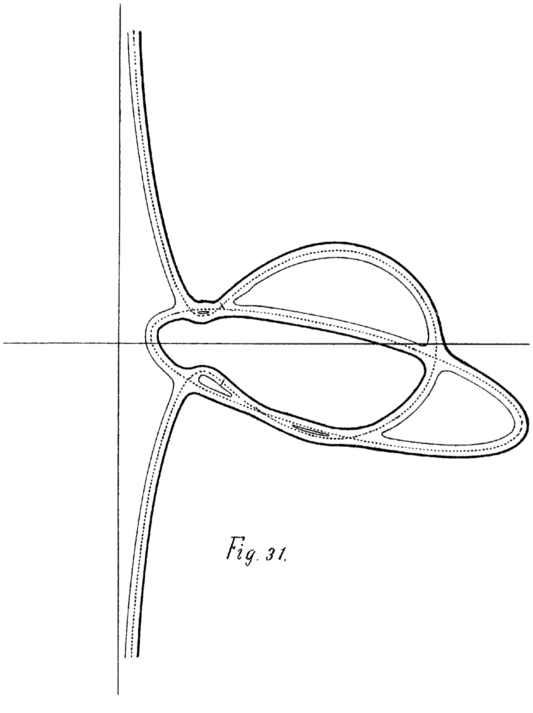

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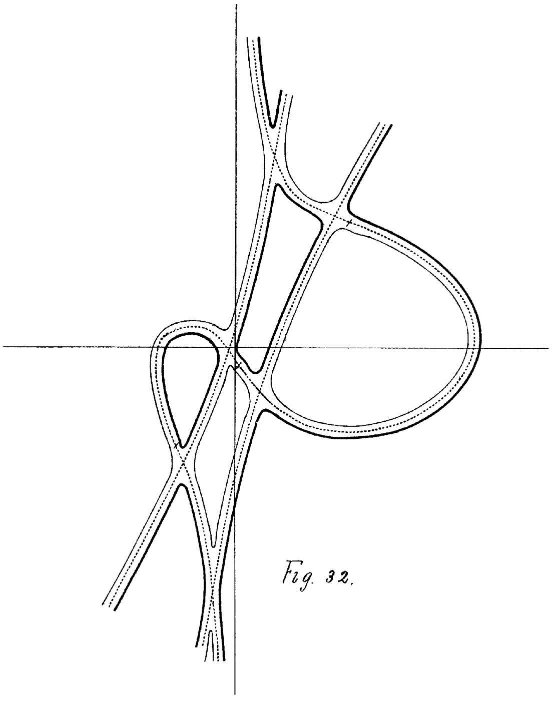

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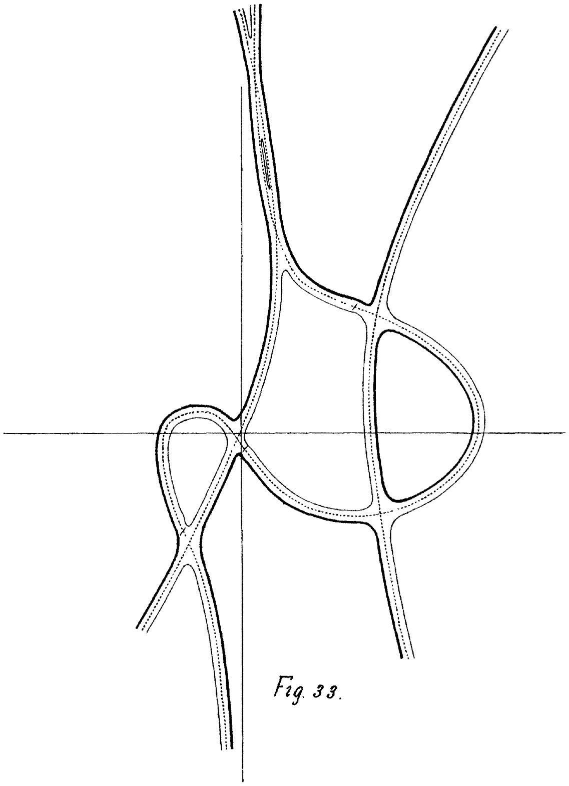

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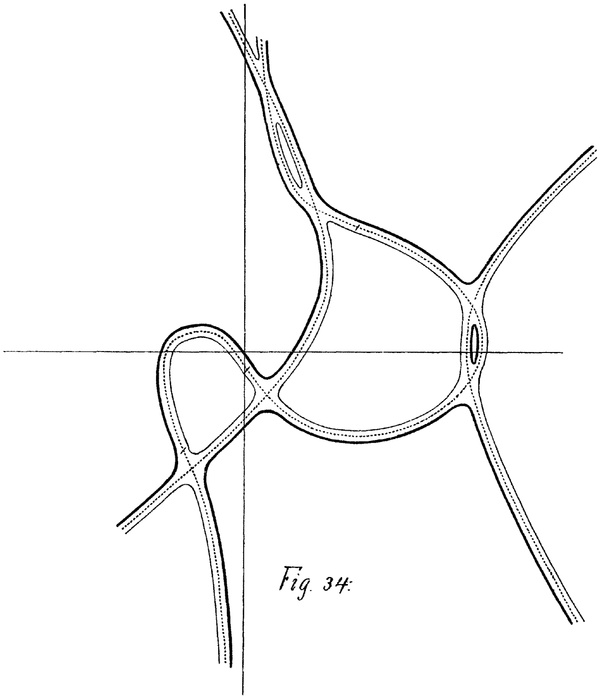

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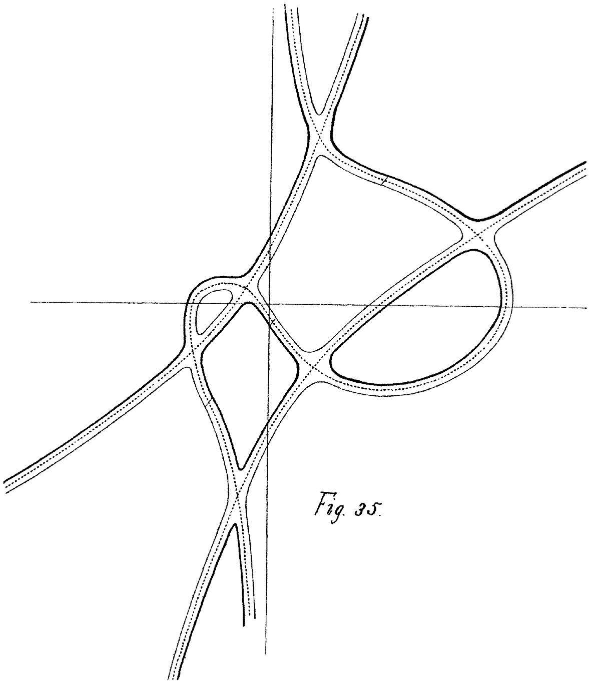

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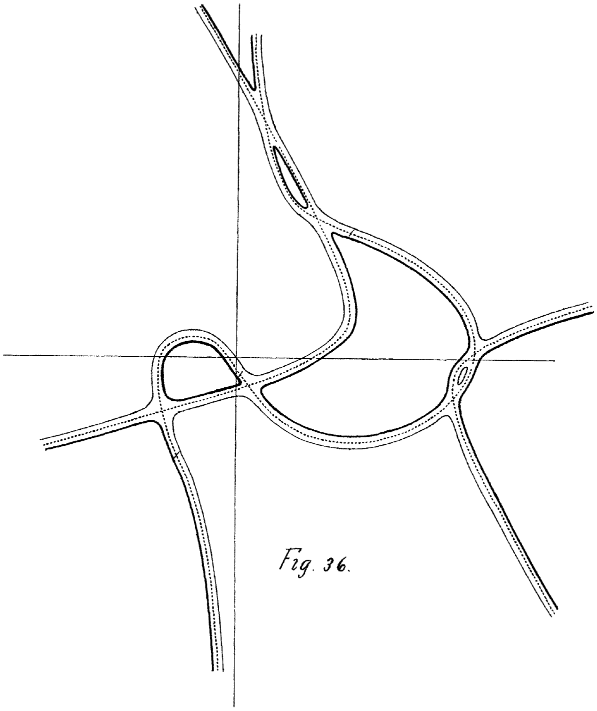

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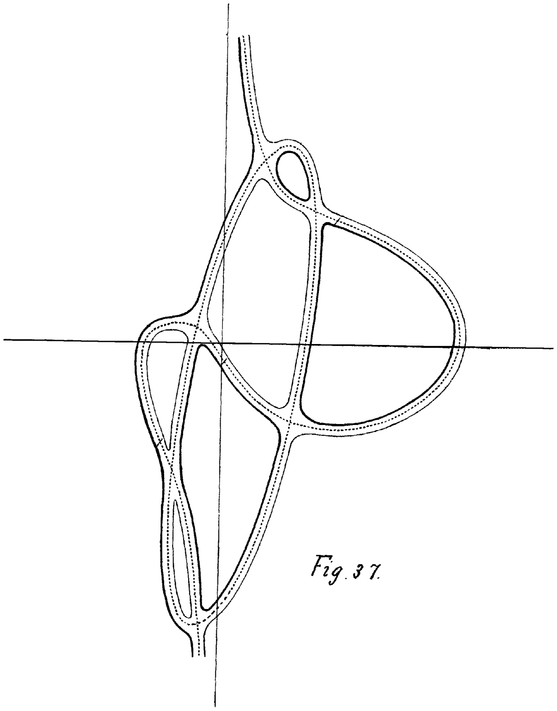

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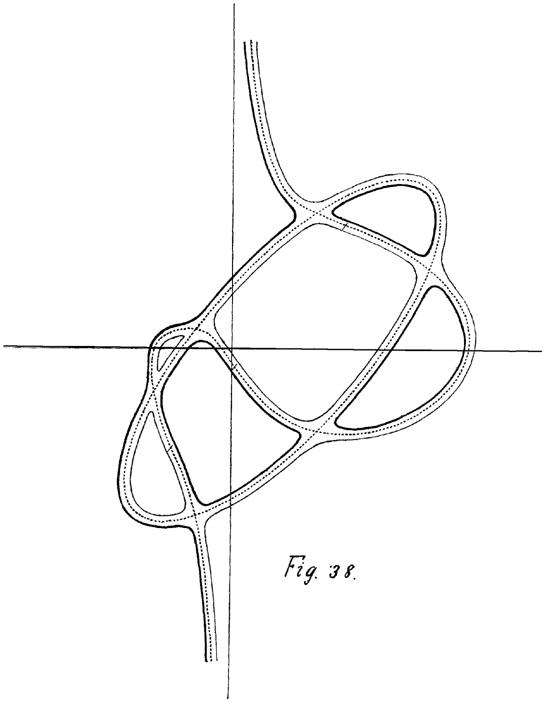

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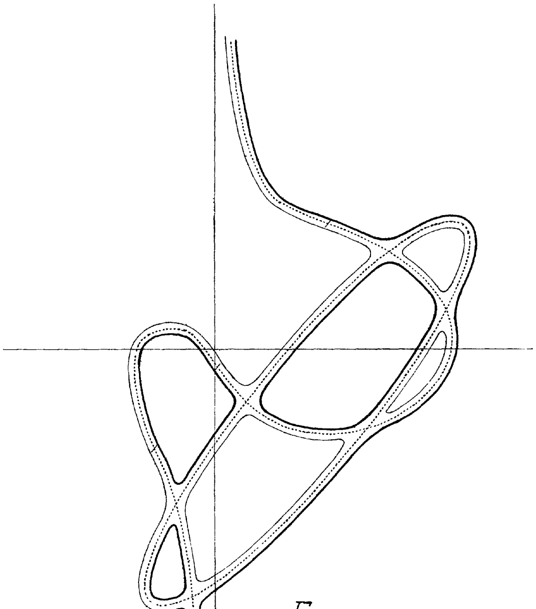

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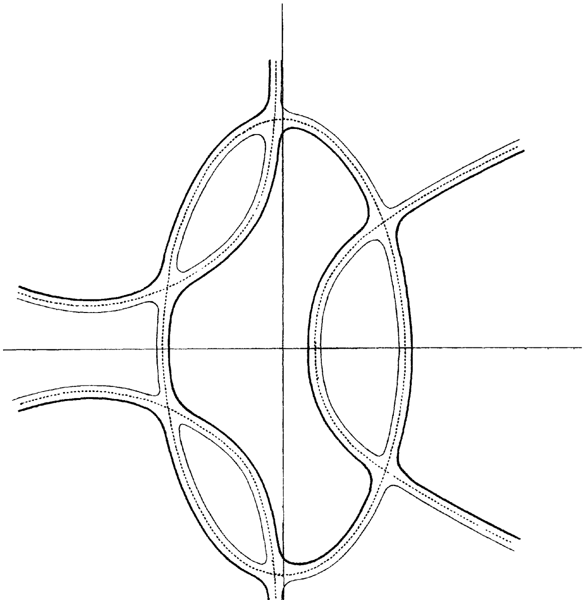

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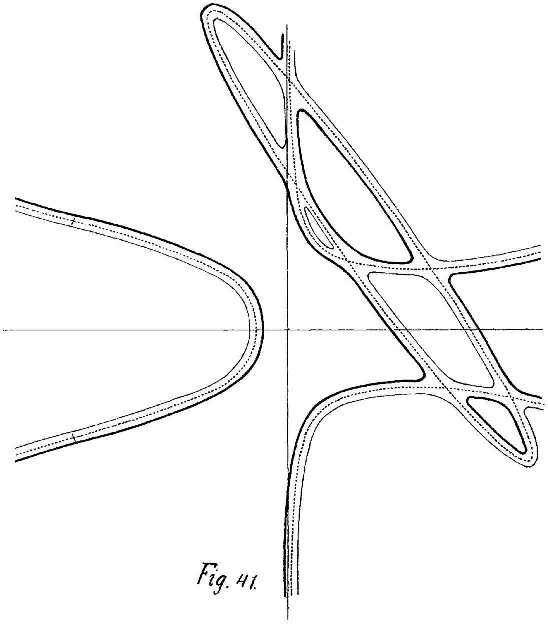

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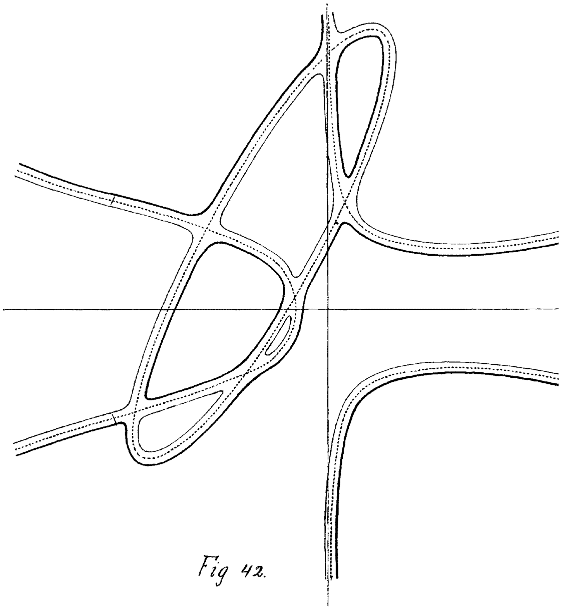

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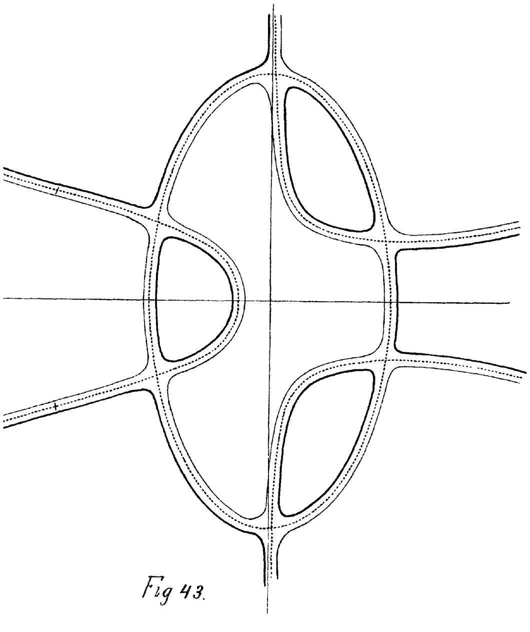

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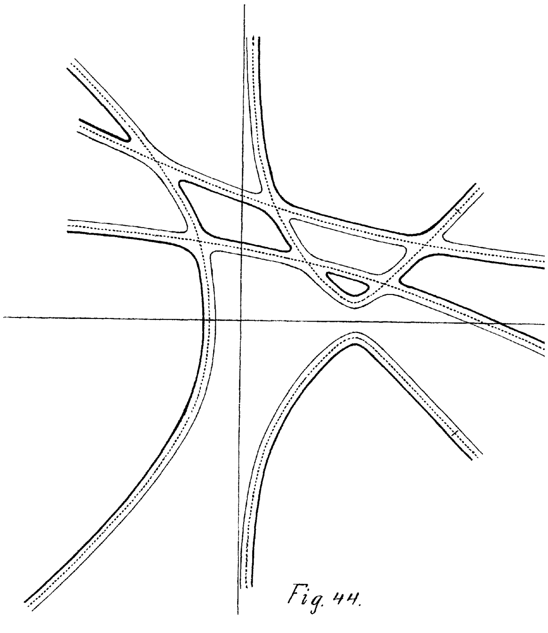

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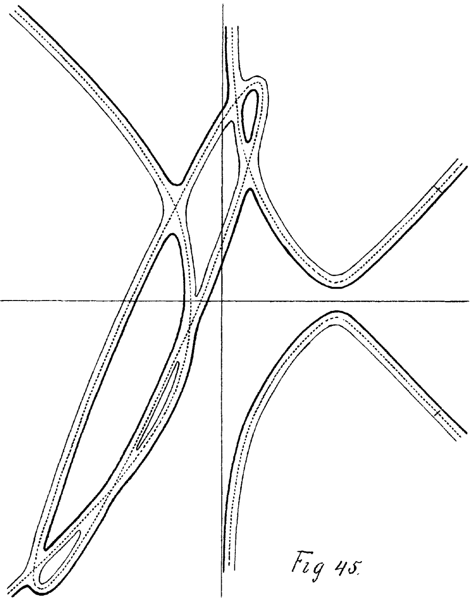

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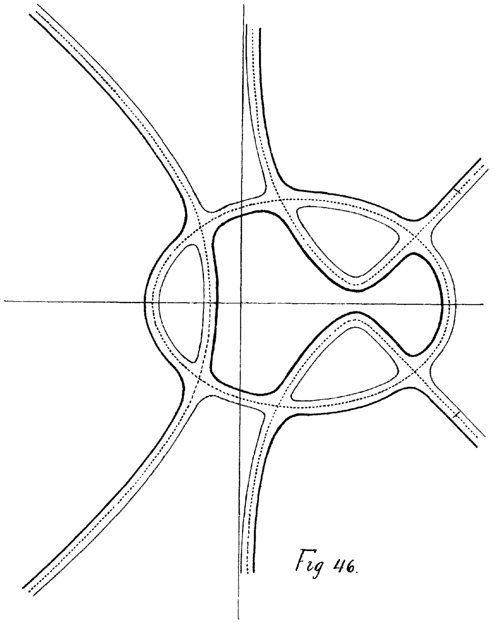

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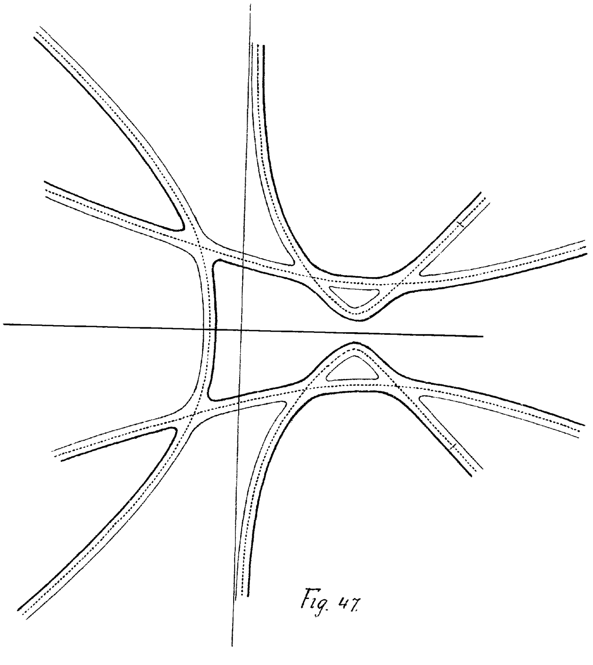

Fig. 47.